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Thermodynamics of the critical $RSOS(q_1, q_2; q)$ model

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Abstract
The thermodynamic Bethe ansatz method is employed for the study of the integrable critical $RSOS(q_1, q_2; q)$ model. The high and low temperature behaviours are investigated, and the central charge of the effective conformal field theory is derived. The obtained central charge is expressed as the sum of the central charges of two generalized coset models.

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1. Introduction

It is well known that statistical systems at criticality—second-order phase transition—are expected to exhibit conformal invariance [1], therefore the critical behaviour of such systems should be described by a certain conformal field theory. Different types of critical behaviour have been classified [2], and the critical exponents and correlation functions have been determined (see also [3, 4]).

An intriguing situation arises from the study of integrable lattice models, whose scaling limit may correspond to certain conformal field theories. In this framework an important, but non-trivial task is the calculation of the central charge of the corresponding conformal field theory. A way one can extract this information is by studying the finite size effects of the ground state of the system [5–7]. An alternative approach to compute the conformal properties is by investigating the low temperature thermodynamics; in particular, the low temperature behaviour of the free energy of a critical system is described by [8, 9]

$$\frac{F(T)}{L} = \frac{F_0}{L} - \frac{\pi c}{6u} T^2 + \cdots \qquad T \ll 1. \tag{1.1}$$

For integrable theories this can be achieved by means of the thermodynamic Bethe ansatz approach, which is a powerful technique that allows the computation of such properties. The mathematical techniques used for such computations go back to the original work of several people [10–15]. The method was further treated and extended to various lattice [16–19] (for a review on TBA for lattice models see e.g. [20]) and continuum relativistic models [21–25] yielding very important results.

The thermodynamic Bethe ansatz for relativistic models is somehow the inverse of the Bethe ansatz technique for lattice models [26–30]. In the usual Bethe ansatz approach the starting point is the microscopic Hamiltonian, whose diagonalization gives rise to the Bethe ansatz equations, the spectrum and the scattering information—expressed via the *S* matrix—(see e.g. [27, 28]). On the other hand, in the integrable relativistic theories one employs the scattering information as an input in order to derive the thermodynamics of the theory [21, 22].

In this study the thermodynamics of the $RSOS(q_1, q_2; q)$ is investigated and the effective conformal anomaly is derived. In general, *RSOS* models are worth studying because, as already mentioned, their critical behaviour may be described by some effective conformal field theory, e.g. critical fused *RSOS* models are related to generalized diagonal coset models ('anti-ferromagnetic' regime) or parafermionic theories ('ferromagnetic' regime) [31]. Furthermore, it has been shown [32] that critical *RSOS* models, with proper inhomogeneities, provide lattice regularizations of massive or massless integrable quantum field theories [32], which on the other hand can be thought of as perturbations of conformal field theories [33]. What makes the $RSOS(q_1, q_2; q)$ model in particular interesting is that it is a natural generalization of the $RSOS(p, q)$ model studied by Bazhanov and Reshetikhin [31] in as much as the alternating spin chain, introduced by de Vega and Woyanorovich [34], is a generalization of the fused *XXZ* spin chain [35]. Therefore, with this paper the study of the thermodynamics of the fused critical *RSOS* models is completed.

In [31] the $RSOS(p, q)$ model was studied, the effective central charge was found and, in the 'anti-ferromagnetic' regime, it turned out to be the one of the $SU(2)$ diagonal coset model $\mathcal{M}(p, \nu-2-p)$ ($\mathcal{M}(q, p) \equiv \frac{SU(2)_q \otimes SU(2)_p}{SU(2)_{q+p}}$, where $SU(2)_k$ is the $SU(2)$ *WZW* model at level k [36, 37]), whereas in the 'ferromagnetic' regime it agreed with the central charge of the parafermionic $\frac{SU(2)_{\nu-2}}{U(1)}$ theory. In this work the effective central charge of the $RSOS(q_1, q_2; q)$ model is computed from the low temperature analysis. In the 'anti-ferromagnetic' regime it is expressed as the sum of the central charges of two generalized diagonal coset models, namely $\mathcal{M}(q_2, \nu-q_2-2)$ and $\mathcal{M}(q_2, \delta q)$, while in the 'ferromagnetic' regime the analysis is exactly the same as in [31].

The outline of this paper is as follows: in the next section the model is introduced, and the Bethe ansatz equations and the energy spectrum are presented. In the third section the thermodynamic Bethe ansatz equations are derived explicitly and the high and low temperature behaviours are examined. Finally, from the low temperature expansion the effective central charge is derived.

2. The model

The integrable critical $RSOS(q_1, q_2; q)$ model, obtained from the $RSOS(1, 1)$ model by fusion [38, 39], is introduced. To describe the model, a square lattice of $2N$ horizontal and M vertical sites is considered. The Boltzmann weights associated with every site are defined as

$$w(l_i, l_j, l_m, l_n|\lambda) \equiv \begin{pmatrix} l_n & l_m \\ l_i & l_j \end{pmatrix}. \tag{2.1}$$

With every face i of the lattice an integer l_i is associated, and every pair of adjacent integers satisfy the following restriction conditions [40, 41]:

$$0 \leqslant l_{i+1} - l_i + P \leqslant 2P \tag{2.2a}$$

$$P \leqslant l_{i+1} + l_i \leqslant 2\nu - P \tag{2.2b}$$

where $P = q_1$ for i odd and $P = q_2$ for i even (let $q_1 > q_2$), for the horizontal pairs, and $P = q$ for the vertical pairs (array type II [32]).

The fused Boltzmann weights have been derived by Date *et al* in [39] and they are given by

$$w^{q_i,1}\left(a_1, a_{q_i+1}, b_{q_i+1}, b_1|\lambda\right) = \sum_{a_2\ldots a_{q_i}} \prod_{k=1}^{q_i} w^{1,1}(a_k, a_{k+1}, b_{k+1}, b_k|\lambda + \mathrm{i}(k - q_i)) \tag{2.3}$$

where $b_2 \ldots b_{q_i}$ are arbitrary numbers satisfying $|b_i - b_{i+1}| = 1$. $w^{1,1}$ are the Boltzmann weights for the $SOS(1, 1)$ model [40], they are non-vanishing as long as the condition (2.2*a*), for $P = 1$ is satisfied and they are given by the following expressions:

$$\begin{aligned} w(l, l \pm 1, l, l \mp 1|\lambda) &= h(\mathrm{i} - \lambda) \\ w(l \pm 1, l, l \mp 1, l|\lambda) &= -h(\lambda)\frac{h_{l+1}}{h_l} \\ w(l \pm 1, l, l \pm 1, l|\lambda) &= h(w_l \pm \lambda)\frac{h_1}{h_l} \end{aligned} \tag{2.4}$$

where

$$h(\lambda) = \rho\Theta(\lambda)H(\lambda). \tag{2.5}$$

$H(\lambda)$ and $\Theta(\lambda)$ are Jacobi theta functions and

$$h_l = h(w_l) \qquad w_l = w_0 + \mathrm{i}l. \tag{2.6}$$

We are interested in the critical case where $h(\lambda)$ becomes a simple trigonometric function, i.e.

$$h(\lambda) = \frac{\sinh \mu\lambda}{\sin \mu} \tag{2.7}$$

w_0, ρ and μ are arbitrary constants. Furthermore,

$$\begin{aligned} w^{q_i,q}(a_1, b_1, b_{q+1}, a_{q+1}) &= \prod_{k=0}^{q-2}\prod_{j=0}^{q_i-1}(h(\mathrm{i}(k - j) + \lambda))^{-1} \\ &\quad\times \sum_{a_2\ldots a_q}\prod_{k=1}^{q} w^{q_i,1}(a_k, b_k, b_{k+1}, a_{k+1}|\lambda + \mathrm{i}(k - 1)) \end{aligned} \tag{2.8}$$

again $b_2 \ldots b_{q_i}$ are arbitrary numbers satisfying $|b_i - b_{i+1}| = 1$, and the pairs a_1, a_{q+1} and b_1, b_{q+1} satisfy (2.2), for $P = q$. The fused weights satisfy the Yang–Baxter equation in the following form:

$$\begin{aligned} &\sum_g w^{pq}(a, b, g, f|\lambda)w^{ps}(f, g, d, e|\lambda + \mu)w^{qs}(g, b, c, d|\mu) \\ &\qquad = \sum_g w^{qs}(f, a, g, e|\mu)w^{ps}(a, b, c, g|\lambda + \mu)w^{pq}(g, c, d, e|\lambda). \end{aligned} \tag{2.9}$$

Here we only need the explicit expressions for $w^{q_i,1}$ which are

$$\begin{aligned} w^{q_i,1}(l + 1, l' + 1, l', l|\lambda)) &= h_{q_i-1}^{q_i-1}(-\lambda)h_a\frac{h(\mathrm{i}b - \lambda)}{h_l} \\ w^{q_i,1}(l + 1, l' - 1, l', l|\lambda)) &= h_{q_i-1}^{q_i-1}(-\lambda)h_b\frac{h(\lambda + \mathrm{i}a)}{h_l} \\ w^{q_i,1}(l - 1, l' + 1, l', l|\lambda)) &= h_{q_i-1}^{q_i-1}(-\lambda)h_c\frac{h(\mathrm{i}d - \lambda)}{h_l} \\ w^{q_i,1}(l - 1, l' - 1, l', l|\lambda)) &= h_{q_i-1}^{q_i-1}(-\lambda)h_d\frac{h(\mathrm{i}c - \lambda)}{h_l} \end{aligned} \tag{2.10}$$

where

$$a = \frac{l + l' - q_i}{2} \qquad b = \frac{l' - l + q_i}{2} \qquad c = \frac{l - l' + q_i}{2} \qquad d = \frac{l + l' + q_i}{2} \tag{2.11}$$

and

$$h_k^q(\lambda) = \prod_{j=0}^{q-1} h(\lambda + \mathrm{i}(k - j)). \tag{2.12}$$

It is obvious that $w^{q_i,1}(a, b, c, d|\lambda)$ are periodic functions, because they involve only simple trigonometric functions (2.10), (2.12) $\left(h(\lambda + \mathrm{i}\nu) = -h(\lambda),\ \nu = \frac{\pi}{\mu}\right)$, i.e.

$$w^{q_i,1}(a, b, c, d|\lambda + \mathrm{i}\nu) = (-)^{q_i} w^{q_i,1}(a, b, c, d|\lambda). \tag{2.13}$$

Now we can define the transfer matrix of the $RSOS(q_1, q_2; q)$ model

$$T^{q_1,q_2;q\{b_1\ldots b_{2N}\}}_{\{a_1\ldots a_{2N}\}} = \prod_{j=1}^{2N-1} w^{q_1,q}(a_j, a_{j+1}, b_{j+1}, b_j|\lambda) w^{q_2,q}(a_{j+1}, a_{j+2}, b_{j+2}, b_{j+1}|\lambda) \tag{2.14}$$

where we impose periodic boundary conditions, i.e. $a_{2N+1} = a_1$ and $b_{2N+1} = b_1$. Note that in the odd and even sites the weights $w^{q_1,q}$ and $w^{q_2,q}$ live, respectively. The case where $q_1 = q_2$ (array type I [32]), namely the fused $RSOS(p, q)$ model, has been studied in detail by Bazhanov and Reshetikhin in [31]. It is evident that the model studied here is a generalization of the fused $RSOS(p, q)$ model. The analogue of the array type II in the spin chain framework is the alternating quantum spin chain, introduced by de Vega and Woyanorovich [34], and also studied extensively by many authors [42–46].

From the Yang–Baxter equation for the fused Boltzmann weights (2.9) the commutativity property for the transfer matrix follows, i.e.

$$T^{q_1,q_2;q}(\lambda) T^{q_1,q_2;q'}(\mu) = T^{q_1,q_2;q'}(\mu) T^{q_1,q_2;q}(\lambda). \tag{2.15}$$

Moreover the transfer matrix is periodic (2.13)

$$T^{q_1,q_2;q}(\lambda + \mathrm{i}\nu) = T^{q_1,q_2;q}(\lambda). \tag{2.16}$$

In order to obtain the Bethe ansatz equations for the model we also need the following useful relations. First we will use the relations acquired by the fusion procedure [31, 39], namely

$$T_0^{q_1,q_2;q} T_q^{q_1,q_2;1} = f_q^{q_1,q_2} T_0^{q_1,q_2;q-1} + f_{q-1}^{q_1,q_2} T_0^{q_1,q_2;q+1} \tag{2.17}$$

where

$$f_q^{q_1,q_2}(\lambda) = \left(h_q^{q_1}(\lambda) h_q^{q_2}(\lambda)\right)^N \qquad T_k^{q_1,q_2;q} = T^{q_1,q_2;q}(\lambda + \mathrm{i}k) \qquad T_0^{q_1,q_2;0} = f_{-1}^{q_1,q_2}. \tag{2.18}$$

Note that the main difference between equations (2.17), (2.18) and the corresponding equations in [31] is the substitution of p with q_1, q_2. In particular f_q^p in [31] is replaced here by $f_q^{q_1,q_2}$. We must also have in mind that the Boltzmann weights satisfy the following important property, i.e. up to a gauge transformation, that does not affect the transfer matrix, the weights $w^{1,q}(a, b, c, d|\lambda)$ and $w^{1,\nu-2-q}(\nu - a, \nu - b, c, d|\lambda + \mathrm{i}(q + 1))$ coincide, where

$$w^{1,q}(a, d, c, b|\lambda - \mathrm{i}(q - 1)) = \left(h_{q-1}^{q-1}(-\lambda)\right)^{-1} w^{q,1}(a, b, c, d|\lambda) \tag{2.19}$$

a similar property holds also between the weights $w^{q_i,q}$ and $w^{q_i,\nu-2-q}$. From the above relations it follows that

$$\begin{aligned} T^{q_1,q_2;q}(\lambda) &= Y T^{q_1,q_2;\nu-2-q}(\lambda + \mathrm{i}(q + 1)) \qquad q = 1, \ldots, \nu - 3 \\ T^{q_1,q_2;\nu-2}(\lambda) &= Y \left(h_{\nu-2}^{q_1}(\lambda) h_{\nu-2}^{q_2}(\lambda)\right)^N \end{aligned} \tag{2.20}$$

with

$$Y^{\{l'_1\ldots l'_{2N}\}}_{\{l_1\ldots l_{2N}\}} = \prod_{i=1}^{2N} \delta(l_i, \nu - l'_i) \qquad [T^{q_1,q_2;q}, Y] = 0. \tag{2.21}$$

To derive the transfer matrix eigenvalues we employ the commutativity properties of the transfer matrix (2.15), (2.21), the periodicity (2.13), (2.16), the fusion relations (2.17), (2.18), equations (2.20) and the analyticity of the eigenvalues. Moreover, we employ relations (2.17) and (2.20) for $q = \nu - 1, \nu$ and we derive

$$T^{q_1,q_2;\nu-1}(\lambda) = 0 \qquad T^{q_1,q_2;\nu}(\lambda) = -Y f^{q_1,q_2}_{\nu-1}(\lambda). \tag{2.22}$$

From the solution of the above system of equations (2.15)–(2.21), and with the help of relations (2.22) we can write equation (2.17) in the following form:

$$\det M[\Lambda^{q_1,q_2;1}(\lambda)] = 0 \tag{2.23}$$

where

$$M[\Lambda^{q_1,q_2;1}(\lambda)] = \begin{pmatrix} \Lambda^{q_1,q_2;1}_0 & f^{q_1,q_2}_{-1} & 0 & 0 & . & 0 & 0 & -Yf^{q_1,q_2}_0 \\ f^{q_1,q_2}_1 & \Lambda^{q_1,q_2;1}_1 & f^{q_1,q_2}_0 & 0 & . & 0 & 0 & 0 \\ 0 & f^{q_1,q_2}_2 & \Lambda^{q_1,q_2;1}_2 & f^{q_1,q_2}_1 & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & f^{q_1,q_2}_{\nu-2} & \Lambda^{q_1,q_2;1}_{\nu-2} & f^{q_1,q_2}_{\nu-3} \\ -Yf^{q_1,q_2}_{\nu-2} & 0 & 0 & 0 & . & 0 & f^{q_1,q_2}_{\nu-1} & \Lambda^{q_1,q_2;1}_{\nu-1} \end{pmatrix}. \tag{2.24}$$

Now let $\left(Q^{q_1,q_2}_0(\lambda), \ldots, Q^{q_1,q_2}_{\nu-1}(\lambda)\right)$ be the null vector of the matrix (2.24) with $Q^{q_1,q_2}_k(\lambda) = \omega^k Q^{q_1,q_2}(\lambda + \mathrm{i}k)$, $\omega^{2\nu} = 1$ and

$$Q^{q_1,q_2}(\lambda) = \prod_{j=1}^{\frac{(q_1+q_2)N}{2}} h(\lambda - \lambda_j) \tag{2.25}$$

then the eigenvalues are given by the following expression:

$$\Lambda^{q_1,q_2;1}(\lambda) = \omega f^{q_1,q_2}_{-1}(\lambda)\frac{Q^{q_1,q_2}(\lambda + \mathrm{i})}{Q^{q_1,q_2}(\lambda)} + \omega^{-1} f^{q_1,q_2}_0(\lambda)\frac{Q^{q_1,q_2}(\lambda - \mathrm{i})}{Q^{q_1,q_2}(\lambda)}. \tag{2.26}$$

For completeness we write the general expression of the eigenvalues $\Lambda^{q_1,q_2;q}(\lambda)$, which follow from the fusion relation (2.17) and (2.26),

$$\Lambda^{q_1,q_2;q}(\lambda) = Q^{q_1,q_2}(\lambda - \mathrm{i})Q^{q_1,q_2}(\lambda + \mathrm{i}q)\sum_{j=0}^{q} \frac{\omega^{q-2j} f^{q_1,q_2}(\lambda + \mathrm{i}(j-1))}{Q^{q_1,q_2}(\lambda + \mathrm{i}(j-1))Q^{q_1,q_2}(\lambda + \mathrm{i}j)}. \tag{2.27}$$

The eigenvalues satisfy all equations (2.17), (2.18) and (2.20), where ω is a root of unity that obeys the constraint

$$\omega^\nu = -(-)^{\frac{(q_1+q_2)N}{2}} y \tag{2.28}$$

and $y = \pm 1$ is the eigenvalue of the operator Y (2.21). Equation (2.28) is a consequence of the periodicity and (2.20). Similarly, here the difference with the corresponding eigenvalues in [31] is the replacement of the functions f^p and Q^p with f^{q_1,q_2} and Q^{q_1,q_2}, respectively. Finally, from the analyticity of the eigenvalues we obtain the Bethe ansatz equations

$$\omega^{-2} e_{q_1}(\lambda_\alpha)^N e_{q_2}(\lambda_\alpha)^N = -\prod_{\beta=1}^{M} e_2(\lambda_\alpha - \lambda_\beta) \tag{2.29}$$

where

$$e_n(\lambda;\nu)=\frac{\sinh\mu\left(\lambda+\frac{\mathrm{i}n}{2}\right)}{\sinh\mu\left(\lambda-\frac{\mathrm{i}n}{2}\right)}. \tag{2.30}$$

It is important to emphasize that the eigenstates of the model are states with zero spin $S_z=0$ [31, 32, 47], i.e.

$$M=\tfrac{1}{4}(q_1+q_2)L \tag{2.31}$$

where $L=2N$ (for $q_1=q_2=p$ the later constraint agrees with the corresponding constraint in [31]). We should mention that the Bethe ansatz equations (2.29) have the same structure with the Bethe ansatz equations of the alternating $\frac{q_1}{2},\frac{q_2}{2}$ spin chain [34–45]. The main differences between the model under study and the alternating spin chain are: (1) the phase ω which is unity, and (2) the number of strings M which is not fixed in the alternating spin chain.

The energy[1] of a state is characterized by the set of quasi-particles with rapidities (Bethe ansatz roots) λ_j, [27, 28, 38],

$$E=-\frac{\mu}{8\pi}\sum_{j=1}^{M}\sum_{n=1}^{2}\frac{\sin\mu q_n}{\sinh\mu\left(\lambda_j+\frac{\mathrm{i}q_n}{2}\right)\sinh\mu\left(\lambda_j-\frac{\mathrm{i}q_n}{2}\right)}. \tag{2.33}$$

The thermodynamic limit $N\to\infty$ of equation (2.29) can be studied with the help of the string hypothesis [12, 13, 27, 28], which states that solutions of (2.29) in the thermodynamic limit are grouped into strings of length n with the same real part and equidistant imaginary parts

$$\lambda_\alpha^{(n,j)}=\lambda_\alpha^n+\frac{\mathrm{i}}{2}(n+1-2j)\qquad j=1,2,\ldots,n$$
$$\lambda_\alpha^{(0,s)}=\lambda_\alpha^0+\mathrm{i}\frac{\pi}{2\mu}$$

where λ_α^n and λ_α^0 are real, and $\lambda_\alpha^{(0,s)}$ is the negative parity string. The allowed strings that describe the thermodynamics of the model are the same as in [31] and they are $1\leqslant n\leqslant\nu-2$ ($q_i\leqslant\nu-2$), the negative parity string is also excluded. Then, the Bethe ansatz equations (2.29) following [12, 13] become

$$\omega^{-2}\prod_{j=1}^{2}X_{nq_j}\left(\lambda_\alpha^n\right)^N=-\prod_{m=1}^{\nu-2}\prod_{\beta=1}^{M_m}E_{nm}\left(\lambda_\alpha^n-\lambda_\beta^m\right) \tag{2.35}$$

where $n=1,\ldots,\nu-2$ and

$$\begin{aligned}X_{nm}(\lambda)&=e_{|n-m+1|}(\lambda)e_{|n-m+3|}(\lambda)\ldots e_{(n+m-3)}(\lambda)e_{(n+m-1)}(\lambda)\\E_{nm}(\lambda)&=e_{|n-m|}(\lambda)e^2_{|n-m+2|}(\lambda)\ldots e^2_{(n+m-2)}(\lambda)e_{(n+m)}(\lambda).\end{aligned} \tag{2.36}$$

Finally, the energy (2.33) by virtue of the string hypothesis (2.34) takes the form

$$E=-\frac{L}{4}\sum_{n=1}^{\nu-2}\int_{-\infty}^{\infty}\mathrm{d}\lambda\left(Z^{(\nu)}_{nq_1}(\lambda)+Z^{(\nu)}_{nq_2}(\lambda)\right)\rho_n(\lambda) \tag{2.37}$$

[1] The Hamiltonian of the model is defined for $q=q_1,q_2$

$$H=-\frac{\mu}{8\pi}\sum_{i=1}^{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln T^{q_1,q_2;q_i}(\lambda)|_{\lambda=0} \tag{2.32}$$

where $T^{q_1,q_2;q_i}$ is the transfer matrix of the $RSOS(q_1,q_2;q_i)$ model (see also (2.27)).

where, ρ_n is the density[2] of the n strings (pseudo-particles) and

$$Z_{nm}^{(\nu)}(\lambda) = \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathrm{i}\log X_{nm}(\lambda) \tag{2.39}$$

the Fourier transform of the last expression is

$$\hat{Z}_{nm}^{(\nu)}(\omega) = \frac{\sinh\left((\nu - \max(n,m))\frac{\omega}{2}\right)\sinh\left(\min(n,m)\frac{\omega}{2}\right)}{\sinh\left(\frac{\nu\omega}{2}\right)\sinh\left(\frac{\omega}{2}\right)}. \tag{2.40}$$

3. Thermodynamic Bethe ansatz

In what follows the thermodynamic Bethe ansatz equations are derived from (2.35). In addition to the density of pseudo-particles ρ_n we also introduce the density of holes $\tilde{\rho}_n$, and we can immediately deduce from (2.35), and with the help of the Maclaurin expansion (2.38) that they satisfy

$$\tilde{\rho}_n(\lambda) = \frac{1}{2}\left(Z_{nq_1}^{(\nu)}(\lambda) + Z_{nq_2}^{(\nu)}(\lambda)\right) - \sum_{m=1}^{\nu-2} A_{nm}^{(\nu)} * \rho_m(\lambda) \tag{3.1}$$

where

$$A_{nm}^{(\nu)}(\lambda) = \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathrm{i}\log E_{nm}(\lambda) + \delta_{nm}\delta(\lambda) \tag{3.2}$$

and

$$\hat{A}_{nm}^{(\nu)}(\omega) = \frac{2\coth\left(\frac{\omega}{2}\right)\sinh\left((\nu - \max(n,m))\frac{\omega}{2}\right)\sinh\left(\min(n,m)\frac{\omega}{2}\right)}{\sinh\left(\frac{\nu\omega}{2}\right)}. \tag{3.3}$$

However, recall that the only allowed states as in [31] are those with $S_z = 0$ and therefore from (2.31),

$$\sum_{n=1}^{\nu-2} n\int_{-\infty}^{\infty}\rho_n(\lambda)\,\mathrm{d}\lambda = \frac{q_1+q_2}{4}. \tag{3.4}$$

Equation (3.4) together with relation (3.1) for $n = \nu - 2$ yields

$$\int_{-\infty}^{\infty}\tilde{\rho}_{\nu-2}(\lambda)\,\mathrm{d}\lambda = 0 \quad\Rightarrow\quad \tilde{\rho}_{\nu-2}(\lambda) = 0. \tag{3.5}$$

The constraint (3.5) is imposed on (3.1) and the density $\rho_{\nu-2}$ is expressed in terms of the rest densities,

$$\rho_{\nu-2}(\lambda) = \rho^0(\lambda) - \sum_{m=1}^{\nu-3} a_{\nu-2-m}^{(\nu-2)} * \rho_m(\lambda) \tag{3.6}$$

where

$$\hat{a}_n^{(\nu-2)}(\omega) = \frac{\sinh\left((\nu-n-2)\frac{\omega}{2}\right)}{\sinh\left((\nu-2)\frac{\omega}{2}\right)} \qquad \hat{\rho}^0(\omega) = \frac{\sinh\left(q_1\frac{\omega}{2}\right) + \sinh\left(q_2\frac{\omega}{2}\right)}{4\cosh\left(\frac{\omega}{2}\right)\sinh\left((\nu-2)\frac{\omega}{2}\right)}. \tag{3.7}$$

[2] Here we use the Maclaurin expansion

$$\sum_{j=1}^{M} f(\lambda_j) \sim L\int_{-\infty}^{\infty} f(\lambda)\rho(\lambda)\,\mathrm{d}\lambda. \tag{2.38}$$

By means of relation (3.6) equation (3.1) can be rewritten in the following form:

$$\tilde{\rho}_n(\lambda) = \frac{1}{2}\big(Z^{(\nu-2)}_{nq_1}(\lambda) + Z^{(\nu-2)}_{nq_2}(\lambda)\big) - \sum_{m=1}^{\nu-3} A^{(\nu-2)}_{nm} * \rho_m(\lambda). \tag{3.8}$$

The energy of the system, after we apply the string hypothesis is given by (2.37). Now, taking into account equation (3.8) the energy becomes

$$e = \frac{E}{L} = -g_0 - \frac{1}{4}\sum_{n=1}^{\nu-3}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \left(Z^{(\nu-2)}_{nq_1}(\lambda) + Z^{(\nu-2)}_{nq_2}(\lambda)\right)\rho_n(\lambda) \tag{3.9}$$

with

$$g_0 = \frac{1}{16\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\left(\sinh\left(q_1\frac{\omega}{2}\right) + \sinh\left(q_2\frac{\omega}{2}\right)\right)^2}{\sinh\left(\frac{\nu\omega}{2}\right)\sinh\left((\nu-2)\frac{\omega}{2}\right)}. \tag{3.10}$$

In order to determine the thermodynamic Bethe ansatz equations the free energy of the system should be minimized, i.e., $\delta F = 0$, where

$$F = E - TS \tag{3.11}$$

and the entropy of the system is given by

$$\begin{aligned} S \simeq L\sum_{n=1}^{\nu-3}\int_{-\infty}^{\infty} \mathrm{d}\lambda((\rho_n(\lambda) + \tilde{\rho}_n(\lambda))\ln(\rho_n(\lambda) + \tilde{\rho}_n(\lambda)) - \rho_n(\lambda)\ln\rho_n(\lambda) - \tilde{\rho}_n(\lambda)\ln\tilde{\rho}_n(\lambda)) \\ = L\sum_{n=1}^{\nu-3}\int_{-\infty}^{\infty} \mathrm{d}\lambda\left(\rho_n(\lambda)\ln\left(1 + \frac{\tilde{\rho}_n(\lambda)}{\rho_n(\lambda)}\right) + \tilde{\rho}_n(\lambda)\ln\left(1 + \frac{\rho_n(\lambda)}{\tilde{\rho}_n(\lambda)}\right)\right). \end{aligned} \tag{3.12}$$

Then, from equations (3.9), (3.12) and the constraint (3.8) the following expression is implied:

$$T\ln(1+\eta_n(\lambda)) = -\frac{1}{4}\big(Z^{(\nu-2)}_{nq_1}(\lambda) + Z^{(\nu-2)}_{nq_2}(\lambda)\big) + \sum_{m=1}^{\nu-3} A^{(\nu-2)}_{nm} * T\ln\left(1 + \eta_m^{-1}(\lambda)\right) \tag{3.13}$$

where $\eta_n(\lambda) = \frac{\tilde{\rho}_n(\lambda)}{\rho_n(\lambda)}$. It is convenient to consider the convolution of expression (3.13) with the inverse of A_{nm},

$$\hat{A}^{-1}_{nm}(\omega) = \delta_{nm} - \hat{s}(\omega)(\delta_{nm+1} + \delta_{nm-1}) \tag{3.14}$$

having in mind the following identity,

$$A^{-1}_{nm} * Z_{mq_i}(\lambda) = s(\lambda)\delta_{nq_i} \tag{3.15}$$

where

$$s(\lambda) = \frac{1}{2\cosh(\pi\lambda)} \qquad \hat{s}(\omega) = \frac{1}{2\cosh\left(\frac{\omega}{2}\right)} \tag{3.16}$$

and $\eta_n(\lambda) = \mathrm{e}^{\frac{\epsilon_n(\lambda)}{T}}$, (3.13) becomes

$$\epsilon_n(\lambda) = s(\lambda) * T\ln(1+\eta_{n+1}(\lambda))(1+\eta_{n-1}(\lambda)) - \tfrac{1}{4}s(\lambda)\left(\delta_{nq_1} + \delta_{nq_2}\right) \tag{3.17}$$

for any $n = 1, \ldots, \nu - 3$. Note that the last equation differs from the corresponding equation obtained in [31] in the inhomogeneity term $s(\lambda)$. More specifically, here the terms δ_{nq_1} and δ_{nq_2} appear, whereas in the study of the fused $RSOS(p, q)$ model [31] only the δ_{np} term appears. It is obvious that for $q_1 = q_2 = p$ our expression agrees with the corresponding expression for the pseudo-energies in [31]. It can be easily deduced from equation (3.17) that the pseudo-energy $\epsilon_n(\lambda) > 0$ for every $n \neq q_1, q_2$, therefore we conclude that the ground state consists of two filled Dirac seas with strings of length q_1, q_2, i.e. $\tilde{\rho}_n(\lambda) = 0$ for any n, and

$\rho_n(\lambda) = 0$ for any $n \neq q_1, q_2$. The pseudo-energies for those are immediately induced from (3.13) by neglecting the terms of the sum for $m \neq q_i$,

$$\epsilon_i(\lambda) = -\frac{1}{4}\sum_{j=1}^{2} Z^{(\nu-2)}_{q_i q_j}(\lambda) + \sum_{j=1}^{2} \tilde{A}^{(\nu-2)}_{q_i q_j} * T \ln\left(1 + \eta^{-1}_{q_j}(\lambda)\right) \qquad i = 1, 2 \tag{3.18}$$

(NB $\epsilon_i(\lambda) \equiv \epsilon_{q_i}(\lambda)$) where

$$\tilde{A}^{(\nu-2)}_{nm}(\lambda) = A^{(\nu-2)}_{nm}(\lambda) - \delta_{nm}\delta(\lambda). \tag{3.19}$$

Moreover, the energy of the ground state can be written from (3.8), (3.9)

$$e_0 = \frac{E_0}{L} = -g_0 - \frac{1}{8}\sum_{i,j=1}^{2}\int_{-\infty}^{\infty} \mathrm{d}\lambda Z^{(\nu-2)}_{q_i q_j}(\lambda)s(\lambda) = -\frac{1}{8}\sum_{i,j=1}^{2}\int_{-\infty}^{\infty} \mathrm{d}\lambda Z^{(\nu)}_{q_i q_j}(\lambda)s(\lambda). \tag{3.20}$$

The free energy of the system follows from (3.9), (3.11), (3.12), (3.8) and (3.13),

$$f(T) = \frac{F(T)}{L} = -g_0 - \frac{T}{2}\sum_{n=1}^{\nu-3}\int_{-\infty}^{\infty} \mathrm{d}\lambda \ln\left(1 + \eta^{-1}_n(\lambda)\right)\left(Z^{(\nu-2)}_{nq_1}(\lambda) + Z^{(\nu-2)}_{nq_2}(\lambda)\right) \tag{3.21}$$

and in terms of the ground-state energy of the system (3.20) we can write

$$f(T) = e_0 - \frac{T}{2}\sum_{i=1}^{2}\int_{-\infty}^{\infty} \mathrm{d}\lambda s(\lambda) \ln\left(1 + \eta_{q_i}(\lambda)\right). \tag{3.22}$$

In the following sections we are going to explore the behaviour of the free energy and the entropy of the system in the high and low temperatures.

3.1. The high temperature expansion

By studying the high temperature behaviour of the entropy the number of states of the model can be deduced. In the high temperature limit the pseudo-energies ϵ_n become independent of λ [18], consequently the thermodynamic Bethe ansatz equations (3.17) are given by

$$\epsilon_n \simeq s(\lambda) * T \ln(1 + \eta_{n+1})(1 + \eta_{n-1}) = \frac{T}{2}\ln(1 + \eta_{n+1})(1 + \eta_{n-1}) \tag{3.23}$$

and the corresponding solution of the above difference equation is exactly the same as in [31] (for $T \to \infty$ the inhomogeneity term can be neglected in (3.17) and therefore the pseudo-energies coincide with those found in [31])

$$\ln(1 + \eta_n) = \ln\frac{\sin^2\left(\frac{\pi(n+1)}{\nu}\right)}{\sin^2\left(\frac{\pi}{\nu}\right)}. \tag{3.24}$$

The free energy follows immediately from (3.22), (3.24)

$$F = -\frac{TL}{4}\sum_{n=q_1,q_2} \ln\frac{\sin^2\left(\frac{\pi(n+1)}{\nu}\right)}{\sin^2\left(\frac{\pi}{\nu}\right)} \tag{3.25}$$

moreover, the entropy in the high temperature limit (3.11) becomes

$$S = \frac{L}{2}\sum_{n=q_1,q_2} \ln\frac{\sin\left(\frac{\pi(n+1)}{\nu}\right)}{\sin\left(\frac{\pi}{\nu}\right)}. \tag{3.26}$$

Note here that the free energy and the entropy are expressed as a sum of two terms since the ground state consists of two filled Dirac seas. On the other hand, in [31] the corresponding expressions contain just one term, because the ground state there consists of one filled Dirac sea. Finally, we conclude that the number of states for the system is

$$\prod_{n=q_1,q_2}\left(\frac{\sin\left(\frac{\pi(n+1)}{\nu}\right)}{\sin\left(\frac{\pi}{\nu}\right)}\right)^{\frac{L}{2}}. \tag{3.27}$$

Note that in the isotropic limit $\nu\to\infty$ the entropy (3.26) coincides with that of the alternating $\frac{q_1}{2}, \frac{q_2}{2}$ spin chain (see e.g. [43, 46]). For $q_1=q_2$ (3.26) agrees with the entropy found in [31].

3.2. The low temperature expansion

The main purpose of this section is the derivation of the effective central charge via the study of the low temperature thermodynamics. Recall, that the ground state of the model consists of two filled Dirac seas of strings q_1, q_2, therefore we examine the TBA (3.13) for $n=q_1, q_2$. In the $T\to 0$ limit the following quantities are defined:

$$T\ln\left(1+\eta_i^{\pm}\right)\to\pm\epsilon_i^{\pm} \qquad i=1,2 \tag{3.28}$$

with

$$\epsilon_i^-=\tfrac{1}{2}(\epsilon_i-|\epsilon_i|) \qquad \epsilon_i^+=\epsilon_i-\epsilon_i^- \tag{3.29}$$

then the pseudo-energies for the ground state (3.18) take the form

$$\epsilon_i(\lambda)=-\frac{1}{4}\sum_{j=1}^{2}Z_{q_iq_j}^{(\nu-2)}(\lambda)-\sum_{j=1}^{2}\tilde{A}_{q_iq_j}^{(\nu-2)}*\epsilon_j^-(\lambda). \tag{3.30}$$

Finally, the last equation can be written in terms of ϵ_i, ϵ_i^+

$$\sum_{j=1}^{2}A_{q_iq_j}^{(\nu-2)}*\epsilon_j(\lambda)=-\frac{1}{4}\sum_{j=1}^{2}Z_{q_iq_j}^{(\nu-2)}(\lambda)+\sum_{j=1}^{2}\tilde{A}_{q_iq_j}^{(\nu-2)}*\epsilon_j^+(\lambda) \tag{3.31}$$

and the solution of the above system is given by the following expression:

$$\epsilon_i(\lambda)=-\frac{1}{4}s(\lambda)+\sum_{j=1}^{2}K_{ij}*\epsilon_j^+(\lambda) \qquad i=1,2 \tag{3.32}$$

where the kernel K is

$$K(\lambda)=\begin{pmatrix}h_1(\lambda) & h(\lambda)\\ h(\lambda) & h_2(\lambda)\end{pmatrix} \tag{3.33}$$

$$\hat{h}_1(\omega)=\frac{\sinh\left((\delta q-1)\frac{\omega}{2}\right)}{2\cosh\left(\frac{\omega}{2}\right)\sinh\left(\delta q\frac{\omega}{2}\right)}+\frac{\sinh\left((\nu-3-q_1)\frac{\omega}{2}\right)}{2\cosh\left(\frac{\omega}{2}\right)\sinh\left((\nu-2-q_1)\frac{\omega}{2}\right)}$$

$$\hat{h}_2(\omega)=\frac{\sinh\left((\delta q-1)\frac{\omega}{2}\right)}{2\cosh\left(\frac{\omega}{2}\right)\sinh\left(\delta q\frac{\omega}{2}\right)}+\frac{\sinh\left((q_2-1)\frac{\omega}{2}\right)}{2\cosh\left(\frac{\omega}{2}\right)\sinh\left(q_2\frac{\omega}{2}\right)} \qquad \hat{h}(\omega)=\frac{\sinh\left(\frac{\omega}{2}\right)}{2\cosh\left(\frac{\omega}{2}\right)\sinh\left(\delta q\frac{\omega}{2}\right)} \tag{3.34}$$

and $\delta q=q_1-q_2$. Note, that the expression of the kernel (3.33), (3.34) in this general form for any q_1, q_2 is rather a new result. As long as the condition $q_1=\nu-2-q_2$ holds, the symmetry between left and right sectors is satisfied (see also e.g. [32]). In particular, $h_1=h_2$, with h_1, h_2 being related to the scattering in the left (right) sector. In general, for $\delta q\neq 1$ each of h_i is decomposed into two parts (see (3.34)), and every part is related to the triplet amplitude of the *XXZ* model, with different anisotropy parameters (hidden degrees of freedom

[38, 46, 48]). In the special case where $\delta q = 1$, there are no hidden degrees of freedom, and h_1, h_2 are relevant to the triplet amplitudes of the *XXZ* (sine-Gordon) model with the proper anisotropy parameters, whereas h corresponds to the massless LR scattering amplitude (see also [45, 49]).

To derive the effective central charge, the entropy of the system must be evaluated in the low temperature limit. In order to do that the following approximations, which hold true for $\lambda \to \infty$, should be made [16–18],

$$\rho_n(\lambda) \simeq \frac{2}{\pi} f_n(\lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda} \epsilon_n(\lambda) \qquad \tilde{\rho}_n(\lambda) \simeq \frac{2}{\pi}(1 - f_n(\lambda)) \frac{\mathrm{d}}{\mathrm{d}\lambda} \epsilon_n(\lambda) \tag{3.35}$$

where $f_n(\lambda) = \left(1 + \mathrm{e}^{\frac{\epsilon_n(\lambda)}{T}}\right)^{-1}$, $(f_0(\lambda) = f_{\nu-2}(\lambda) \equiv 1)$ and the entropy (3.12), can be written as

$$s = \frac{S}{L} = -\frac{2}{\pi} \sum_{n=1}^{\nu-3} \int_{\epsilon_n(-\infty)}^{\epsilon_n(\infty)} \mathrm{d}\epsilon_n (f_n(\lambda) \ln f_n(\lambda) + (1 - f_n(\lambda)) \ln(1 - f_n(\lambda))). \tag{3.36}$$

By changing variables in the last expression,

$$s = \frac{2T}{\pi} \sum_{n=1}^{\nu-3} \int_{f_n^{\min}}^{f_n^{\max}} \mathrm{d}f_n \left(\frac{\ln f_n}{1 - f_n} + \frac{\ln(1 - f_n)}{f_n} \right) \tag{3.37}$$

and by introducing the Rogers dilogarithm

$$L(x) = -\frac{1}{2} \int_0^x \mathrm{d}y \left(\frac{\ln y}{1 - y} + \frac{\ln(1 - y)}{y} \right) \tag{3.38}$$

the entropy can be written in terms of the dilogarithms as follows:

$$s = -\frac{4T}{\pi} \sum_{n=1}^{\nu-3} \left(L\left(f_n^{\max}\right) - L\left(f_n^{\min}\right) \right). \tag{3.39}$$

The next natural step is the solution of the TBA equations (3.17) in the low temperature limit. In order to do that it is convenient (see also [16–18, 31]) to introduce the function

$$\phi_n(\lambda) = \frac{1}{T} \epsilon_n \left(\lambda - \frac{1}{\pi} \ln T \right) \tag{3.40}$$

then the TBA equations become

$$\phi_n \simeq -s(\lambda) * \ln f_{n+1} f_{n-1} - \tfrac{1}{4}\, \mathrm{e}^{-\pi\lambda} \left(\delta_{nq_1} + \delta_{nq_2} \right). \tag{3.41}$$

Our task is to solve the later difference equation in the limit that $\lambda \to \pm\infty$, (ϕ_n independent of λ). First for $\lambda \to \infty$ we compute the $f_n^{\max}$, the difference equations (3.41) become

$$\phi_n \simeq -\tfrac{1}{2} \ln f_{n+1} f_{n-1} \qquad n = 1, \ldots, \nu - 3 \tag{3.42}$$

this system has been solved (see e.g. [18, 31]) with the solution being (note again that the inhomogeneity term is omitted),

$$f_n^{\max} = \frac{\sin^2\left(\frac{\pi}{\nu}\right)}{\sin^2\left(\frac{\pi(n+1)}{\nu}\right)} \qquad n = 1, \ldots, \nu - 3. \tag{3.43}$$

Similarly, for $\lambda \to -\infty$

$$\begin{aligned} &\phi_n \simeq -\tfrac{1}{2} \ln f_{n+1} f_{n-1} && n = 1, \ldots, \nu - 3 \qquad n \neq q_1, q_2 \\ &\phi_{q_1} \to -\infty && \phi_{q_2} \to -\infty \end{aligned} \tag{3.44}$$

the solution of the later system has the following form:

$$f_n^{\min} = \frac{\sin^2\left(\frac{\pi}{q_2+2}\right)}{\sin^2\left(\frac{\pi(n+1)}{q_2+2}\right)} \qquad n = 1, \ldots, q_2 - 1 \qquad f_{q_2}^{\min} = 1$$
$$f_n^{\min} = \frac{\sin^2\left(\frac{\pi}{q_1-q_2+2}\right)}{\sin^2\left(\frac{\pi(n-q_2+1)}{q_1-q_2+2}\right)} \qquad n = q_2 + 1, \ldots, q_1 - 1 \qquad f_{q_1}^{\min} = 1 \quad (3.45)$$
$$f_n^{\min} = \frac{\sin^2\left(\frac{\pi}{\nu-q_1}\right)}{\sin^2\left(\frac{\pi(n-q_1+1)}{\nu-q_1}\right)} \qquad n = q_1 + 1, \ldots, \nu - 3.$$

Note that the main difference with the corresponding solution in [31] is the appearance of the middle term in (3.45) (for $n = q_2+1, \ldots, q_1-1$), in [31] there is no such term in the solution since $q_1 = q_2 = p$. According to equation (3.39) and the above solutions, the entropy can be written as

$$s = -\frac{4T}{\pi}\sum_{n=2}^{\nu-2}\left\{L\left(\frac{\sin^2\left(\frac{\pi}{\nu}\right)}{\sin^2\left(\frac{\pi n}{\nu}\right)}\right) - \sum_{n=2}^{q_2} L\left(\frac{\sin^2\left(\frac{\pi}{q_2+2}\right)}{\sin^2\left(\frac{\pi n}{q_2+2}\right)}\right) - 2L(1)\right.$$
$$\left. - \sum_{n=2}^{q_1-q_2} L\left(\frac{\sin^2\left(\frac{\pi}{q_1-q_2+2}\right)}{\sin^2\left(\frac{\pi n}{q_1-q_2+2}\right)}\right) - \sum_{n=2}^{\nu-q_1-2} L\left(\frac{\sin^2\left(\frac{\pi}{\nu-q_1}\right)}{\sin^2\left(\frac{\pi n}{\nu-q_1}\right)}\right)\right\}. \qquad (3.46)$$

Moreover,

$$\sum_{n=2}^{q-2} L\left(\frac{\sin^2\left(\frac{\pi}{q}\right)}{\sin^2\left(\frac{\pi n}{q}\right)}\right) = \frac{2(q-3)}{q}L(1) \qquad q > 3 \qquad (3.47)$$

and $L(1) = \frac{\pi^2}{6}$ (see e.g. [31]), then

$$s = \frac{2\pi T}{3}\left(\frac{3q_2}{q_2+2} + \frac{3\delta q}{\delta q + 2} - \frac{6q_1}{\nu(\nu - q_1)}\right). \qquad (3.48)$$

The knowledge of the entropy allows the calculation of the heat capacity, in particular

$$C_u = T\frac{\partial s(T)}{\partial T} = -T\frac{\partial^2 f(T)}{\partial^2 T} \qquad (3.49)$$

also, at low temperature it has been shown that [8, 9],

$$C_u = \frac{\pi c}{3u}T + \cdots \qquad (3.50)$$

where c is the central charge of the effective conformal field theory, and u is the speed of sound (Fermi velocity). By means of (3.48), (3.49) and (3.50) ($u = \frac{1}{2}$ in our notation, see e.g. [27]) we can readily deduce the central charge

$$c = \frac{3q_2}{q_2+2} + \frac{3\delta q}{\delta q + 2} - \frac{6q_1}{\nu(\nu - q_1)}. \qquad (3.51)$$

Recall the LR symmetry condition $q_1 = \nu - 2 - q_2$, then the conformal anomaly can be expressed in terms of q_2 and ν as

$$c = \frac{3q_2}{q_2+2} - \frac{6q_2}{\nu(\nu - q_2)} + \frac{3q_2}{q_2+2} - \frac{6q_2}{\tilde{\nu}(\tilde{\nu} - q_2)} \qquad (3.52)$$

where $\tilde{\nu} = \nu - q_2$. Note that the later expression is written in terms of the central charges of two copies of the generalized $SU(2)$ diagonal coset theory. More specifically, the conformal anomaly (3.52) is identified as the sum of the central charges of the $\mathcal{M}(q_2, \nu - q_2 - 2)$ and

$\mathcal{M}(q_2, \tilde{\nu} - q_2 - 2) \equiv \mathcal{M}(q_2, \delta q)$ coset models, therefore the effective conformal field theory should be of the form $\mathcal{M}(q_2, \nu - q_2 - 2) \otimes \mathcal{M}(q_2, \delta q)$.

Expression (3.51) for $q_1 = q_2$ is compatible with the result obtained by Bazhanov and Reshetikhin—in the 'anti-ferromagnetic' regime[3]—in [31]. In the special case where $q_2 = 1$, the central charge becomes

$$c = 2 - \frac{12}{\nu(\nu-2)} = 1 - \frac{6}{\nu(\nu-1)} + 1 - \frac{6}{(\nu-1)(\nu-2)} \tag{3.53}$$

and it agrees with the c_{IR} presented in [32], given by the sum of the central charges of two unitary minimal models. Finally, in the isotropic limit the central charge (3.51) reduces to that of the alternating $\frac{q_1}{2}, \frac{q_2}{2}$ quantum spin chain (see e.g. [42, 46]).

4. Discussion

The thermodynamics of the critical $RSOS(q_1, q_2; q)$ model, obtained by fusion, was studied and the high and low temperature expansions were discussed. The main result of this work was the derivation of the effective conformal anomaly (3.51), (3.52) of the model, the validity of which was confirmed by various tests. More specifically, for $q_2 = 1$ expression (3.52) coincides with the c_{IR} presented in [32], and it is specified by the sum of the central charges of the unitary minimal models $\mathcal{M}_\nu$, $\mathcal{M}_{\nu-1}$, where

$$c = 1 - \frac{6}{\nu(\nu-1)} \tag{4.1}$$

is the central charge of the unitary minimal model $\mathcal{M}_\nu$ of conformal field theory [2]. Also, in the case where $q_1 = q_2$ we recover the results of [31]. Finally, in the isotropic limit $\nu \to \infty$ our result agrees with the conjectured central charge for the alternating spin chain [42], expressed as the sum of the central charges of $SU(2)_{q_2}$, $SU(2)_{\delta q}$, i.e.

$$c = \frac{3q_2}{q_2+2} + \frac{3\delta q}{\delta q + 2}. \tag{4.2}$$

An exact calculation of the effective central charge for the alternating spin chain, by means of the finite size effects and the thermodynamic Bethe ansatz analysis, is presented in [46]. In general, the central charge (3.52) obtained in the present study is identified as the sum of the central charges of the $\mathcal{M}(q_2, \nu - q_2 - 2)$ and $\mathcal{M}(q_2, \delta q)$ coset models, whereas in [31] Bazhanov and Reshetikhin, by studying the $RSOS(p, q)$ models, found an effective central charge that corresponds to the $\mathcal{M}(p, \nu - p - 2)$ model. We conclude that the effective conformal field theory that emanates from the study of the $RSOS(q_1, q_2; q)$ model, consists of two copies of the generalized $SU(2)$ coset theory.

A compelling task is to extend the above calculations in the presence of boundaries, and compute the boundary energy of the system as well as the corresponding g-function (see e.g. [51–53]). There exist solutions of the boundary Yang–Baxter equation [54] in the *RSOS* representation [55–57], and moreover, in [55] the Bethe ansatz equations of the *RSOS* model with boundaries have been explicitly derived. Finally, a very challenging problem is the formulation of a string hypothesis for integrable critical models associated with non-simply laced algebras such as the $A_2^{(2)}$ (Izergin–Korepin) quantum spin chain [58]. Such a formulation is necessary for the investigation of the thermodynamics as well as the conformal properties of these systems.

[3] The analysis of the 'ferromagnetic' regime is exactly the same as in [31], and it gives rise to the central charge of the parafermionic $\frac{SU(2)_{\nu-2}}{U(1)}$ theory, i.e. $c = 2 - \frac{6}{\nu}$, [50].

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